

Reduze – Feynman Integral Reduction in C++

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ABSTRACT: **Reduze** is a computer program for reducing Feynman Integrals to master integrals employing a Laporta algorithm. The program is written in C++ and uses classes provided by the GiNaC library to perform the simplifications of the algebraic prefactors in the system of equations. **Reduze** offers the possibility to run reductions in parallel.

KEYWORDS: Reduction, Laporta Algorithm, Feynman Integral, Loop Calculation, GiNaC.

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Contents

1. Introduction	2
1.1 License	2
2. Theoretical background	3
2.1 Propagators, Sectors and Integrals	3
2.2 Integration By Parts (IBP) Identities	4
2.3 Lorentz Invariance (LI) Identities	4
2.4 Symmetry Relations	5
2.5 Zero Sectors	5
2.6 Reduction	5
3. Reduction Algorithm	6
4. Usage	7
4.1 Finding an Auxiliary Topology	7
4.2 Reduction	8
4.2.1 Set up an Auxiliary Topology	8
4.2.2 Prepare a Reduction	8
4.2.3 Run the Reduction	9
4.3 Manipulating the Results	9
4.3.1 Select solutions	10
4.3.2 Convert to FORM and Mathematica format	10
5. Input Files	10
5.1 Auxiliary Topology Input File	10
5.2 Reduction Input File	13
5.2.1 Options for the FORM and Mathematica output	14
6. Installation	15
6.1 Prerequisites	15
6.2 Building <code>Reduze</code>	15
7. Performance	16
8. Applications	16

1. Introduction

The calculation of loop amplitudes in perturbative quantum field theory is usually done by generating the Feynman diagrams for the desired physical process and interfering the corresponding analytical expressions, working out the Dirac and/or color structure. The amplitude is then a sum of many dimensionally regularized integrals [1] which have to be computed. These integrals are not independent of each other but related by the Integration by Parts (IBP) Identities [2, 3] and the Lorentz Invariance (LI) Identities [4]. These identities form a homogeneous system of linear equations with the integrals as unknowns and algebraic prefactors which are rational polynomials in the kinematic invariants and the dimension. Using the IBP and LI identities one can express most of the integrals in terms of a small set of integrals, called master integrals.

These identities also exist for phase-space integrals after replacing the delta functions by a difference of propagators with an opposite sign prescription of the imaginary part [6].

The procedure of solving this system of equations is called a reduction. Since one often has to solve systems with thousands of equations, computers have to be used and because the prefactors in front of each integral are rational polynomials, computer algebra systems become indispensable.

Reduze is a computer program written in C++ which generates the IBP and optionally the LI identities and then reduces the integrals to master integrals. **Reduze** uses the **GiNaC** library [8] to perform the simplification of the prefactors.

The reduction algorithm is a Laporta algorithm [5] which is essentially the Gauss algorithm with additional rules to determine the next equation which should be solved and inserted into the others. To get the reduction of a certain Feynman diagram one first defines a set of integrals by restricting the exponents of the propagators. **Reduze** then generates the identities from this set and starts solving the system of equations.

For a reduction of several diagrams **Reduze** can treat different diagrams (with the same number of propagators) simultaneously. One defines how many cores or processors are available and then **Reduze** will automatically launch some reductions simultaneously. The more cores there are available the more diagrams one can reduce in parallel.

Other published reduction programs are **AIR** [11] and **FIRE** [12]. **AIR** is a **Maple** package that implements the Laporta algorithm. **FIRE** is a **Mathematica** package that implements the Laporta algorithm and also a method that uses techniques from Gröbner basis calculations. Then there are also several other private codes.

The advantage of **Reduze** is that it is completely open source, has a low memory footprint and can do reductions in parallel.

1.1 License

The package **Reduze** is Copyright © 2009 Cedric Studerus. This program is free software: you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or (at your option) any later version.

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2. Theoretical background

2.1 Propagators, Sectors and Integrals

A propagator P with momentum flow q and mass m is the expression $q^2 - m^2$ where q^2 denotes the Minkowski scalar product with the metric in the convention $g = \text{diag}(1, -1, -1, -1)$. The momentum q is a linear combination of loop momenta k_i and external momenta p_i .

An auxiliary topology is an ordered set of propagators $A_n = \{P_1, \dots, P_n\}$ such that all scalar products $k_i k_j$ and $k_i p_j$ containing at least one loop momentum k_i can be expressed by a linear combination of propagators from A_n . The auxiliary topology is called an l -loop auxiliary topology if there are l different loop momenta appearing in the momenta q of the propagators. Denoting the number of independent external momenta by m , the auxiliary topology must contain exactly $l(l+1)/2 + lm$ propagators where the first term counts the scalar products between loop momenta only and the second term the products involving both loop and external momenta.

Every subset of t propagators of a given auxiliary topology A_n defines a sector T_t with an unique identification number ID . Physically relevant sectors which correspond to diagrams are also called topologies. Assuming the sector has the propagators P_{j_1}, \dots, P_{j_t} with $\{j_1, \dots, j_t\} \subset \{1, \dots, n\}$, then its identification number is defined as

$$ID = \sum_{k=1}^t 2^{j_k-1}. \quad (2.1)$$

There are in general $\binom{n}{t}$ different t -propagator sectors T_t and $\sum_{t=1}^n \binom{n}{t} = 2^n - 1$ sectors that one can build out of an auxiliary topology A_n . Their identification numbers fulfill $1 \leq ID \leq 2^n - 1$.

A sub-sector T_{t-1} of a sector T_t is a sector where one propagator is removed. There are in general t different sub-sectors for a sector T_t . The sub-sector tree of a sector T_t is the set of all sub-sectors of T_t and recursively all sub-sectors of all these sub-sectors. All sectors of an auxiliary topology A_n , which is the main sector, are in the sub-sector tree of A_n .

To every t -propagator sector T_t with propagators P_{j_1}, \dots, P_{j_t} belongs a infinite set of d -dimensionally regularized l -loop integrals [1] which all share the same propagators in the denominator of the integrand. They have the generic form

$$\int d^d k_1 \dots \int d^d k_l \frac{P_{j_{t+1}}^{s_1} \dots P_{j_n}^{s_{n-t}}}{P_{j_1}^{r_1} \dots P_{j_t}^{r_t}} \quad (2.2)$$

with integer exponents $r_i \geq 1$ and $s_i \geq 0$. In **Reduze** such an integral is represented by a vector $v = \{v_1, \dots, v_n\}$ containing the exponents of the propagators or, more precisely,

$$\text{INT}[t, r, s, ID, \{v_1, \dots, v_n\}] \quad (2.3)$$

where $r = \sum_{i=1}^t r_i \geq t$ is the sum of the propagators in the denominator and $s = \sum_{i=1}^{n-t} s_i \geq 0$ is the sum of the propagators in the numerator. The value v_i is the exponent of propagator $P_i \in A_n$. It is positive if P_i is in the denominator, negative if P_i comes with a (positive) exponent in the numerator and zero if the propagator is not present. The numbers t, r, s as well as the identification number ID of the sector, to which the integral belongs, can be calculated from the vector v .

Consider an n -propagator auxiliary topology A_n with a t -propagator sector T_t . The number of integrals that one can build for certain values of r and s is given by

$$\mathcal{N}(n, t, r, s) = \binom{r-1}{t-1} \binom{s+n-t-1}{n-t-1}. \quad (2.4)$$

The two binomial factors count all possible ways to arrange the exponents of the propagators in the denominator and numerator, respectively.

2.2 Integration By Parts (IBP) Identities

In dimensional regularization [1] the integral over a total derivative is zero. Let \mathbf{I}' be the integrand of an integral of the form (2.2). Then, working out the differentiation in

$$\int d^d k_i \frac{\partial}{\partial k_i^\mu} [q^\mu \mathbf{I}'(p_1, \dots, p_m, k_1, \dots, k_l)] = 0 \quad (2.5)$$

leads to the integration by parts (IBP) identities [2, 3]. The momentum q is an arbitrary loop or external momentum. The index μ is summed over but the index i is not. If there are l loop momenta and m independent external momenta one can therefore build $l(l+m)$ equations from one integral (the seed integral).

2.3 Lorentz Invariance (LI) Identities

One can also use the Lorentz Invariance of the integrals [4]. Taking an integral $\mathbf{I}(p_1, \dots, p_m)$ the following equation holds

$$\sum_{n=1}^m \left(p_n^\nu \frac{\partial}{\partial p_{n\mu}} - p_n^\mu \frac{\partial}{\partial p_{n\nu}} \right) \mathbf{I}(p_1, \dots, p_m) = 0. \quad (2.6)$$

The derivatives can be shifted directly to the integrand of the integral \mathbf{I} . This equation can be contracted with all possible antisymmetric combinations of the external momenta, e.g. $p_{1\mu} p_{2\nu} - p_{1\nu} p_{2\mu}$, which leads to $m(m-1)/2$ equations where m denotes the number of independent external momenta. As it was shown in [7] the LIs do not give new linear independent equations in addition to the IBPs. However, they can accelerate the convergence in a reduction, since in general an LI identity generated from one seed integral cannot be reproduced with the IBP identities generated from the same seed integral alone. **Reduze** offers the possibility to use the LIs.

2.4 Symmetry Relations

Often there are relations between integrals coming from symmetries which can lead to an identification of integrals and even whole sectors.

All integrals are invariant under permutations of the loop momenta and translations of a loop momentum with other momenta. Such a transformation can be used to transform an integral I to an equivalent integral I' and leads to the identity $I = I'$. However, since the integrals are expressed with propagators of an auxiliary topology, the transformations that actually can be used must leave the set of propagators of the auxiliary topology invariant. This means that such a symmetry transformation must lead to a permutation of the propagators and then two equivalent integrals differ only in a permutation of the propagator exponents.

A permutation of the propagator exponents of an integral can alter the sector (sector identification number) it belongs to. This is then valid for all integrals of this sector and one can completely get rid of one of the sectors. These sectors correspond to the same topology.

Sometimes the integrals of an auxiliary topology are invariant under the permutation of external momenta but this permutation leads to propagators which are not contained in the auxiliary topology. One then has to find a transformation on the propagators which transforms the propagators back into the auxiliary topology.

In **Reduze** one can explicitly declare some transformation rules which leave the integrals invariant. Since only transformations that lead to a permutation of propagators can be used, they must be given as permutations. According to these declarations **Reduze** automatically identifies equivalent integrals in the system of equations and also only uses one of them to generate the equations.

2.5 Zero Sectors

It is possible that a whole sector is zero which means that all integrals belonging to this sector are zero. In **Reduze** a sector is set to zero if the solutions of all IBP identities generated from the integral I of this sector with $r = t$ and $s = 0$ (no additional propagators in the numerator and denominator) contain the equation $I = 0$.

2.6 Reduction

To reduce a sector up to $r = r_{max}$ and $s = s_{max}$ means solving the homogeneous system of linear equations which have been built with the IBP and/or LI identities out of the integrals of this sector that have $r \in [r_{min}, r_{max}]$ and $s \in [s_{min}, s_{max}]$. It is not possible to solve the system completely since the rank of every finite set of equations is smaller than the number of unknown integrals it contains. However, the aim of the reduction is to express most of the integrals by a linear combination of only a few (less complicated) integrals, the so-called master integrals.

In a typical reduction of a t -propagator sector one takes the smallest possible values $r_{min} = t$ and $s_{min} = 0$. If one then generates the equations from all integrals with $r \in [r_{min}, r_{max}]$ and $s \in [s_{min}, s_{max}]$ and reduces this system of equations, one usually gets the

solutions for all the integrals that have been chosen to build the system, meaning that all these integrals are expressed as a linear combination of some master integrals.

The equations built to reduce a sector can contain not only integrals of the sector itself but also integrals from sub-sectors. If one tries to reduce the system of equations, the results then still depend on a lot of unsolved integrals of the sub-sectors which also have to be reduced by solving a system of equations built with integrals from the sub-sector considered. Sub-sectors appear as long as the number of propagators does not go below a minimal value.

3. Reduction Algorithm

For a reduction of a certain sector including its sub-sectors **Reduze** first determines the whole sub-sector tree, then reduces the sub-sectors with the smallest number of propagators, inserts the results in the equations of the sectors that depend on these sub-sectors, reduces these sectors, continuing until the desired sector with the largest number of propagators can be reduced. For a reduction of a single sector **Reduze** first will generate the equations and insert the results of the sub-sectors into them if they are available in the default **results**-directory.

Reduze is not able to find a solution for a single integral of a sector, instead it employs all integrals in a user-defined range of r and s for building the equations and then reduces the whole system.

Since the reduction of a single sector often involves a huge amount of equations, the system must be divided into smaller parts. Dividing the equations in subsets, each generated from a set of integrals for a certain value for r and s , can still lead to systems which are too big. In **Reduze** the system of equations is divided into smaller sets with a default number of equations. The number of equations per set can be adjusted by the user. To decide which equation goes in which set, the equations are first sorted in descending order with respect to the most complicated integral each equation contains and then simply divided into smaller sets and stored in temporary files.

The reduction of a sector is now done by first loading the temporary file containing the simplest equations. Then this system is reduced according to the algorithm below and the results are inserted in all the other files. Then the equations from the second file will be reduced and the results again inserted in all other files. This procedure is done for all files. Finally, the results from the temporary files are collected and saved in a single result file.

To give a precise meaning for the instance that an integral is simpler or less complicated than another integral a lexicographic ordering can be defined [5]: For an integral

$$I = \text{INT}[t, r, s, ID, \{v_1, \dots, v_n\}] \quad (3.1)$$

take the vector $\tilde{v}_I = \{t, r, s, ID, v_1, \dots, v_n\}$ with length $n + 4$. Then for two integrals I and J the comparison $I < J$ is true if and only if there exists an $m \in \{1, \dots, n + 4\}$ such that $\tilde{v}_I[m] < \tilde{v}_J[m]$ and $\tilde{v}_I[k] = \tilde{v}_J[k]$ for all $k \in \{1, \dots, m - 1\}$.

This operator can be extended to equations: Equation e_1 is less complicated than equation e_2 , $e_1 < e_2$, if the most complicated integral of e_1 is less complicated than the most complicated integral of e_2 . For a set L of equations that contains the integral I in at least one equation the set $[I]_L$ is defined as the subset of L such that all equations have I as the most complicated integral. In addition, the set $(I)_L$ is defined as the subset of equations of $[I]_L$ with the smallest number of integrals. This set is the subset of L which contains the shortest equations and all equations contain the most complicated integral I as their own most complicated integral. For an equation e the expression $*e$ denotes the same equation but solved for its most complicated integral.

The first part of the reduction algorithm (see Algorithm 1) is to bring the system in a triangular form and the second part of the algorithm is the back substitution.

Algorithm 1 Reduction

```

 $L = \{e_1, \dots, e_n\}$  // list of equations.
 $I = \{I_1, \dots, I_m\}$  // all integrals in  $L$ , sorted in descending order.
 $S = \{\}$  // empty list.
// Triangularization:
for  $i = 1$  to  $m$  do
  choose  $e \in (I_i)_L$ 
   $L = L \setminus e$ 
  replace matching integrals in  $L$  by the r.h.s of  $*e$ 
   $S = \{*e, S\}$ 
end for
//  $S = \{S(1), \dots, S(l)\}$  contains now  $l$  equations sorted in ascending order.
// Back substitution:
for  $i = 1$  to  $l$  do
   $e = S(i)$ 
  for  $j = i + 1$  to  $l$  do
    replace matching integrals in  $S(j)$  by r.h.s of  $*e$ 
  end for
end for

```

4. Usage

4.1 Finding an Auxiliary Topology

Before a reduction can be launched the user has to find an appropriate auxiliary topology which contains the topologies from given Feynman diagrams as sub-sectors. The auxiliary topology should cover as many diagrams as possible and also allows for as many as possible symmetry relations in order to minimize the number of sectors which finally have to be reduced (see section 2.4). The diagrams under consideration need a certain maximal amount of propagators. By building these propagators one has some freedom how to choose the momentum flow and, if the number of propagators of the diagram is smaller than the

number of propagators needed for the auxiliary topology, one must introduce additional auxiliary propagators. This freedom of choosing how the propagators exactly look like and which additional propagators are introduced can be used to set up an auxiliary topology that has as many as possible symmetries.

4.2 Reduction

Since **Reduze** reads and writes a lot to files, it should be run on the local hard disc. For the following it is assumed that the current working directory is a local directory. A reduction is now done in three steps: set up an auxiliary topology, prepare the reduction and run the reduction.

4.2.1 Set up an Auxiliary Topology

First one has to define an auxiliary topology, where momenta, propagators, etc. are declared. For this one creates a file with suffix `.in`, e.g. `topoA.in`, with all the inputs needed. The syntax of this input file is described in section (5.1). There are also some input files in the `example` directory of the package. Then one sets up the auxiliary topology (`topoA`) with the command

```
reduze --setup topoA.in
```

This creates the directory `topoA` and initializes the auxiliary topology. It derives the rules to express scalar products by propagators, finds the sectors that are equivalent due to symmetry relations and finds most of the zero sectors by doing a small reduction ($r = t$, $s = 0$) of all sectors which have vanishing sub-sectors only.

The new created directory `topoA` contains the directories `reduction` and `results` and a log file `setup.log`. The log file contains some information about the auxiliary topology, e.g. at the end of this file there is a list of all sectors that have not been found to be zero and can be reduced. The `reduction` directory will later be used for the reduction. It contains the directory `include` in which the information about the auxiliary topology is stored. One should not edit it directly. If the setup file `topoA.in` was modified, the setup has to be done again. The `results` directory is used for the results after a reduction has completed.

4.2.2 Prepare a Reduction

To prepare a reduction one needs a second input file, e.g. `prepareA.in`, in that one writes which sectors (*ID* numbers) should be reduced and which class of integrals ($R2 = r_{max}$, $S2 = s_{max}$) one wants to use for building the system of equations. Also the maximum number of processes that will run in parallel can be defined here. The allowed commands of this file are described in section (5.2). There are also some example files in the `example` directory. Typing

```
reduze --prepare prepareA.in
      --auxtop /path/to/topoA
```

should evaluate very quickly. It initializes the input data as well as the inputs of the auxiliary topology `topoA`, checks them for consistency and creates the script `run.sh`. Every time one changes this input file it must be reprocessed with `reduze --prepare`. The script `run.sh` will then be overwritten. The option `--auxtop` followed by an absolute or relative path to the directory of the auxiliary topology which is going to be used can be omitted if one already works in the directory of this auxiliary topology.

The input data for the reduction are copied to the directory `reduction/include`. Avoid editing them directly.

4.2.3 Run the Reduction

The last step is the reduction. To start it one launches

```
./run.sh
```

This script starts the reduction of the sectors declared in the input file `prepareA.in` and controls the number of processes running in parallel. For every sector that has to be reduced a new directory will be created in the directory `reduction` with the identification number of the sector as name. This directory is used for log files and temporary data. When a reduction is complete the results are written to the directory `results`. If there are already results for this sector they will be overwritten.

The script `run.sh` copies the executable `reduze` from the installation directory and renames it to `reduzeID`, where ID is the identification number of the sector that has to be reduced. If one kills this script with

```
killall run.sh
```

the reductions of the sectors that have been started already will continue, but no further reductions are started anymore. Single reduction processes can be killed with

```
killall reduzeID
```

where ID corresponds to the sector number.

4.3 Manipulating the Results

`Reduze` writes the results of a reduction in an internal format which is not well suited for further processing with another algebra system. Also, if symmetry relations are used, the result-files only contain the solutions for one of all equivalent integrals. To get all the desired solutions the user must provide a list of integrals for which the solutions should be extracted or generated due to symmetry relations. Then these results can be converted to a `Mathematica`- or `FORM`-readable format [9, 10].

To do this, `Reduze` offers the options `--select`, `--FORM` and `--MMA`. These options should be used together with the option `--auxtop` to tell `Reduze` which auxiliary topology is going to be used.

4.3.1 Select solutions

Reduze can pick the solutions for some user defined integrals from all the results generated during the reduction. Launching

```
reduze --select MyIntegrals
```

where **MyIntegrals** is a file containing a semicolon separated list of the integrals for which one wants to extract the solution. The format of the integrals is the same as in formula (2.3), however, the numbers t , r , s and ID can be omitted or replaced by any other character, which does not match the bracket '{'. A minimalistic example looks like

```
INT[{v1, ..., vn}];
```

The output is written to the file **MyIntegrals.sol**. Integrals for which no solution has been found (e.g. master integrals) will be written to the file **MyIntegrals.rest**.

4.3.2 Convert to FORM and Mathematica format

Invoking **reduze** with the options

```
reduze --FORMAT <files>
```

where **--FORMAT** is either **--FORM** or **--MMA** converts results from **Reduze** format to a **FORM** or **Mathematica** readable format respectively. The parameters given in **<files>** are interpreted as the names of files containing results in the format of **Reduze**. For each input file a new output file is created in the current working directory with the extension **.inc** and **.m** respectively. Usually, one first uses **reduze --select** to extract only the results that one actually needs and then converts them to the desired output format.

The **Mathematica** format is a list of rules and the **FORM**-format is a table of **id** statements. Positive integer powers of non-integer denominators in the prefactors of the **FORM** output are written with the function **Den**, where **Den(a)^k** equals to a^{-k} .

Often one wants to expand the results, or more precisely the prefactors, in a Laurent series around $d - 4$. With **Reduze** one can do this expansion on the fly when creating the **Mathematica** and **FORM** outputs. If such an expansion is desired, the user has to declare this in the reduction input file. The syntax is explained in section (5.2.1).

5. Input Files

5.1 Auxiliary Topology Input File

The input file, which defines an auxiliary topology, is read in to the main program at run time. It must have the suffix **.in**. The input file consists of lines starting with a keyword followed by some values and a semicolon. Keywords and values are separated by white spaces or tabulators. Comments that will not be interpreted by **Reduze** are entered after a double slash and empty lines are ignored.

```
Keyword value; // don't forget the ';'
```

In the following the keywords are explained using an explicit example. The first declarations define the symbols and must be written at the very beginning of the input file.

```
LoopMomenta k1 k2;
ExtMomenta p1 p2 p3;
Symbol s t m;
Dimension d;
```

The keyword **LoopMomenta** defines the loop momenta k_1 and k_2 , **ExtMomenta** the external momenta p_1 , p_2 , p_3 and **Symbols** declares the symbols s , t and m . **Dimension d** sets the name of the dimension d ; however, if this line is missing the default is also taken to be d .

The keyword **Propagator** takes as the first parameter the momentum flow and as the second parameter the mass (not squared). In **Reduze** a propagator with momentum q and mass m is the term $q^2 - m^2$. If one wants to work with the convention of the metric $g = \text{diag}(-1, 1, 1, 1)$, and therefore wants to have a propagator $q^2 + m^2$ one has to replace the mass m by $I*m$.

```
Propagator k1 0;
Propagator k2 0;
Propagator k1-k2 0;
Propagator k1-p1 0;
Propagator k2-p1 0;
Propagator k1-p1-p2 0;
Propagator k2-p1-p2 0;
Propagator k1-p3 m;
Propagator k2-p3 m;
```

These lines define nine propagators of which two have the mass m . The order of the declarations is the same order that will be used in the integral representation $\text{INT}[t, r, s, ID, \{v_1, \dots, v_n\}]$, where v_i is the exponent of the i -th propagator declared above. There must be nine propagators because there are nine scalar products containing loop momenta: $k_i^2, k_1 k_2, k_i p_j$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. If these scalar products cannot uniquely be replaced by propagators **Reduze** will abort.

For diagrams with cuts only the (sub-)sectors which still contain all cut propagators in the denominator of the integrand are non-zero. In other words: for diagrams with cuts one wants to set to zero all integrals, which either miss one or more cut propagators completely or where a cut propagator appears in the numerator of the integrand only. To achieve this one declares the cut propagators with the keyword **CutPropagator** instead of **Propagator**.

With the keyword **Kinematic** all Lorentz scalar products between the external momenta must be replaced by algebraic expressions of the the symbols defined above, otherwise **Reduze** will abort.

```
Kinematic p1^2 = 0;
Kinematic p2^2 = 0;
Kinematic p3^2 = m^2;
```

```

Kinematic p1*p2 = s/2;
Kinematic p2*p3 = (s+t-m^2)/2;
Kinematic p1*p3 = (m^2-t)/2;

```

Reduze will also abort if the left hand side of the equations is not a product of exactly two external momenta or not a squared external momentum or if the right hand side still contains some loop or external momenta. For the process of two incoming massless particles with momenta p_1 and p_2 to two outgoing massive particles with momenta p_3 and $p_4 = p_1 + p_2 - p_3$ with squared mass $p_3^2 = p_4^2 = m^2$ the scalar product between different momenta are replaced by the Mandelstam invariants $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$ in this example.

There are several additional options one can give. One can set a variable, e.g. the mass m , equal to one.

```

SetToOne m;

```

to reduce the number of variables in the rational polynomials by one.

Some integrals with different propagators are equal because the integrands only differ by a shift of a combination of loop/external momenta or a permutation of loop momenta. Those transformations that leave the set of propagators of the auxiliary sector invariant can be given explicitly. The notation is adapted from the cycle representation of permutations of the n propagators and starts with the keyword **Permutation** followed by a sequence of the keyword **Cycle** or **Cyc** with values from $\{1, \dots, n\}$.

```

Permutation Cyc 1 6 Cyc 2 7;
Permutation Cyc 1 2 Cyc 4 5 Cyc 6 7 Cyc 8 9;

```

The first line declares that there is a symmetry transformation which transforms propagator 1 into 6, 6 into 1, and 2 into 7 as well as 7 into 2. This transformation is the translation $k_i \rightarrow -k_i + p_1 + p_2$, $i = 1, 2$, together with the exchange of the external momenta p_1 with p_2 and p_3 with $p_4 = p_1 + p_2 - p_3$. The exchange of the external momenta is allowed since it does not alter the Mandelstam invariants s and t on which the integral actually depends. The second line implements the invariance of interchanging k_1 with k_2 . **Reduze** automatically finds all combinations of the user defined permutations. In the above case the combination of the two permutations

```

Permutation Cyc 1 7 Cyc 2 6 Cyc 4 5 Cyc 8 9;

```

will automatically be added.

The setup procedure automatically determines the zero sectors. Setting the

```

SetupFindZeros true;

```

on **false** will turn this off. User defined zero or non-zero sectors can be declared explicitly with the keywords

```

SetupZeroTopos    ID1 ID2    IDn;
SetupNonZeroTopos ID1 ID2    IDn;

```

where the keywords are followed by the desired sector identification numbers.

5.2 Reduction Input File

To prepare a reduction one creates a file in the directory of the auxiliary topology. It must have the suffix `.in`. The following minimal declarations are necessary to reduce single sectors, where all integrals with $t = R1 \leq r \leq R2$ and $0 = S1 \leq s \leq S2$ are used to generate the equations. This is usually enough to find all solutions of this class of integrals, but the results will still depend on integrals of sub-sectors if they have not been reduced before already.

```
ReduceID 387 385 384 182;
R2 5;
S2 1;
```

The keyword `ReduceID` can be followed by as many sector identification numbers one wants to reduce, the ordering of the numbers plays no role. `Reduce` always begins reducing the sectors with the smallest number t of propagators. It automatically identifies sub-sectors and looks in the directory `results` for solutions for the sub-sectors. If such solutions exist, they will be inserted in the equations of the sector before the reduction starts.

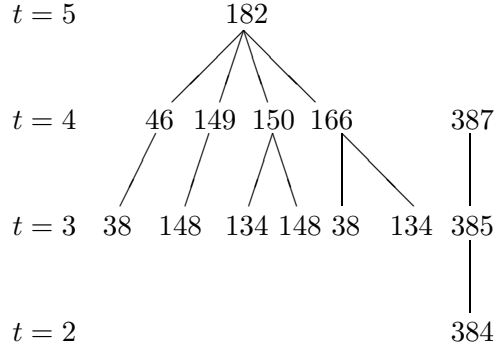


Figure 1: Sub-sector trees of sectors 182 and 387

If one wants to reduce some sectors together with their whole trees of sub-sectors they depend on, one can use the keyword `ReduceIDRecursive` followed by the identification numbers. E.g. for the 5-propagator sector 182 and the 4-propagator sector 387, see Figure (1), one simply writes

```
ReduceIDRecursive 182 387;
```

This will first reduce the 2-propagator sector 384 (sub-sector of sector 385), then the 3-propagator sectors 38, 134, 148 (from the sub-sector tree of sector 182) as well as sector 385, then the 4-propagator sub-sectors of sector 182 and sector 387 and finally sector 182. In this example the two sub-sector trees of 182 and 387 do not overlap, but even if they do, each sector is only reduced once.

If one wants to exclude some sectors from the reduction, for example sector 166, which is a sub-sector of sector 182, it can be done with

```
ReduceIDNot 166;
```

Note that the two sub-sectors (38 and 134) of sector 166 still will be reduced.

Excluding a whole sub-sector tree is done with

```
ReduceIDRecursiveNot 166;
```

This will exclude sector 166 and its whole sub-sector tree, in this case sector 38 and 134, from the reduction.

The commands `ReduceID` and `ReduceIDRecursive` for including some sectors as well as `ReduceIDNot` and `ReduceIDRecursiveNot` for excluding some sectors can have as many sectors as arguments as one wants. One can also combine and repeat them as one likes, but one should notice that if a sector is excluded with one of the exclusion commands it cannot be added again with another command. Adding a sector multiple times has no more effect than adding it once.

There is also the possibility to set $R1$ and $S1$ to another value than t and 0 respectively. However, one cannot write $R1 = t$, since t depends on the sector. Therefore the definition $R1 = 0$ (the default) is interpreted as $R1 = t$ with the t of the sector under consideration, but any other declaration $R1 = n$ with $n > 0$ is taken literally.

The equations build from the integrals can be chosen with

```
UseIBP true;
UseLI false;
```

With this setting, the default, only the IBPs are used.

If the computer used for the reduction has several processors then one can declare

```
Processes n;
```

to tell `Reduze` the number of processes it can run simultaneously. Then, if a reduction is launched with more than one sector with the same t , `Reduze` attempts to run n sectors in parallel.

The system of equations can become quite large, causing memory swapping. To avoid this the system of equations is stored in several files, each holding a certain number of equations, and then only the equations of a single file are loaded into memory and reduced. With the following command one can change the number of the equations per file

```
NrofEqperFile n;
```

where $n = 500$ is the default.

5.2.1 Options for the FORM and Mathematica output

In generating the `FORM` and `Mathematica` results one has the possibility to expand the coefficients in front of each integral in a Laurent series around $\epsilon = 0$, where ϵ is usually defined by $d = 4 - 2\epsilon$. One has to tell `Reduze` the name of ϵ and the relation to the dimension d . The following two commands are the default, if omitted.

```
Epsilon ep;
DimensionRule d = 4-2*ep;
```

The next command tells **Reduze** actually to do this expansion.

```
Series n;
```

The integer n is the order up to which the coefficients should be expanded, including the order n term. To generate the output see section (4.3.2).

6. Installation

6.1 Prerequisites

Reduze uses the **GiNaC** library [8] for the algebraic manipulations. One must install **GiNaC** version 1.4.1 or higher. If the user's Linux distribution provides a compiled package of **GiNaC**, one simply can install the library and headers with the package manager. Usually, if one wants to use a newer version of **GiNaC**, it has to be compiled and installed by hand.

6.2 Building Reduze

The most recent version of **Reduze** can be found at <http://www.itp.uzh.ch/~cedric/reduze/>. Uncompress the package with

```
tar -xzf reduze-version.tgz
```

where **version** is a placeholder for the current version of **Reduze**. Change to the directory **reduze-version** and configure, build, check and install **Reduze** with the commands

```
./configure --prefix=/path/to/inst
make
make check
make install
```

The **--prefix** option can be given if one wants to install **Reduze** in **/path/to/inst/bin** rather than in the default **/usr/local/bin**. The command **make check** sets up an auxiliary topology, does a short reduction and checks if these results are correct by comparing with an internal result file.

The installation directory must be appended to the **PATH** environment variable. If **bash** is used, one writes in the profile file **.bashrc**

```
export PATH=/path/to/inst/bin:$PATH
```

Reduze then can be invoked by typing

```
reduze -h
```

which gives a list of options.

7. Performance

Because of internal memory reasons the maximal number of propagators is limited to $N = 16$. The sum of the exponents of the propagators in the numerator and denominator are limited to $R = 16$ and $S = 8$ respectively.

In a reduction most of the time is used for the algebraic manipulations on the prefactors of the system of equations which are rational polynomials. Any additional variable in these polynomials results in larger expressions and makes the calculations slower. It is therefore very useful to put one scale to one.

Reduze is implemented in a way that from a given set of integrals all IBP identities are generated and then the whole system is reduced. This system contains a lot of redundant equations, which lead to equations $0 = 0$ during the reduction. If the auxiliary topology allows for symmetry transformations then one cannot only get rid of whole sectors but also some integrals in a specific sector can be identified. This can drastically reduce the number of equations.

8. Applications

The program **Reduze** was used to calculate the leading color coefficient and the fermionic corrections to top-quark pair production in the quark-antiquark channel at NNLO [13, 14]. For the leading color coefficient one needs the reduction of the two planar box diagrams in Figure (2). These are four-point functions depending on the Mandelstam invariants s, t , the top-mass m and the dimension d . Including all the sub-sectors there are 60 sectors from which the reduction identities of integrals up to three propagators in the numerator are needed. These about 78'000 integrals are used to generate about half a million IBP identities. The reduction of these equations then gives the solutions of all the 78'000 integrals in terms of 35 master integrals. The running time for this reduction using 10 processors (2300 MHz) takes about 30 hours.



Figure 2: 2-loop boxes for the process $q\bar{q} \rightarrow t\bar{t}$.

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